

F = ma, Important Equation, Big Mistake

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The equation $F=ma$ (Force = mass x acceleration) is considered the most important scientific concept in physics, engineering and, even, in history. It is generally believed that Sir Isaac Newton developed the equation in his Second Law of Motion. In this paper, it is shown that Newton did not originate the equation $F=ma$ and that the Second Law has nothing to do with the equation. The original derivation of the equation is also given.

In his highly acclaimed book, “Mathematical Principles of Natural Philosophy,” or Principia, Newton included the famous Three Laws of Motion. The Second Law states, “*The alteration of motion is ever proportional to the motive force impressed.*” Taking these words to imply the important equation $F=ma$ is perhaps the biggest mistake in the history of science.

“Alteration,” in the Second Law is generally taken to mean the rate of change, or the derivative, of a thing with respect to time. “Force” is believed to be the force of pushing or pulling. “Motion” is the key “key word” in the Second Law. Everyone believes that, here, Newton means the “quantity of motion” of Descartes, which we call today momentum, or mass times velocity, or mv . The equation, $F=ma$, is nowadays derived from Newton’s words by taking the derivative of the momentum with respect to time.

Newton, like other 17th century scientists, used the word “force” freely to refer to our present day force, momentum, energy and power. Right on Page 1 of his book, he defines momentum (Definition III) as the “vis insita,” or “innate force,” or, from his earlier English writings, “*ye force of ye bodys motion.*” This is the “force of a body’s motion” that Descartes used to analyze impact. If the momentum is the “vis insita,” or inertial force, then what is the “quantity of motion,” also defined on Page 1 (Definition II) and used by Newton in the Second Law?

The “motion” in Newton’s Second Law is not Descartes’ quantity of motion, or momentum. Newton’s “motion” is the quantity mv^2 , which is known by the names, “vis viva,” living force, or Leibniz’s Number! The “vis viva” is not hidden in the deep obscurity of the Principia. The vis viva is in plain view, on Page 1 of the book. After defining “the quantity of motion” as “arising from the velocity and quantity of matter (mass) conjointly,” Newton elaborates as follows: “*The motion of the whole is the sum of the motions of all the parts; and therefore in a body double in*

quantity, with equal velocity, the motion is double; with twice the velocity, it is quadruple,” emphasis added.

What happens to the “momentum,” mv , and the “vis viva,” mv^2 , when the mass, m , then the velocity, v , are doubled? Newton’s underlined words are illustrated in the following Tables.

	2m	2v
momentum (mv)	$(2m)(v) = 2mv$	$(m)(2v) = 2mv$
vis viva (mv^2)	$(2m)(v^2) = 2mv^2$	$(m)(\underline{2v})^2 = \underline{4mv^2}$

In words, the Table would look like this:

	2m	2v
momentum (mv)	Doubles	Doubles
vis viva (mv^2)	Doubles	Quadruples

There is one condition when the quantity of motion quadruples, and that is the case of the “vis viva” when the velocity, v , is doubled.

It seems that Newton’s Second Law relates to Leibniz’s Number and not to Descartes’ Motion. To differentiate his “motion” from Leibniz’s motion, Newton did a strange thing in the Principia; he deleted the mass, m , from the expression mv^2 and represented the motion with the remaining term, “ v^2 .” This is shown later with straightforward Examples from the Principia.

First, it is useful to explain why Newton did what he did. Consider the following events:

1. Newton began to work on the Principia in late 1684. The minutes of meetings of the Royal Society (RS) indicate that Halley was told to inform Newton that the RS would protect Newton’s priority to his invention and would publish the anticipated book.
2. In late 1685, Halley told the RS, in response to inquiry, that the book would be finished in about four week.
3. Four weeks passed, then four months, then nine months, and then a year, and the RS did not receive the book. Eventually, the RS told Halley that, if there were a book, he should pay for its publication himself.
4. Finally, the Principia was published in the summer of 1687, 18-months behind schedule.

What happened in early 1686 to delay publication of the Principia and to send Newton into, the widely publicized, feverish activity for 18-months to rewrite the book? Consider the following relevant facts:

1. Before 1686, there was only one “quantity of motion” known in Europe, i.e., Descartes’ momentum, or mv . Let us call Descartes’ motion QM1.
2. In early 1686, Leibniz published, “A Brief Demonstration of a Memorable Error by Descartes.” In this paper, Leibniz introduced another quantity of motion, mv^2 , which he called the “vis viva” or living force.

At the time, Leibniz’s 1686 paper seemed to demolish Descartes’ QM1 as the true measure of “motion.” Newton could not use QM1 for motion in a 1686-87 publication, i.e., after Leibniz’s paper, without first countering Leibniz’s contentions. It took Leibniz ten years to recognize the role of momentum and energy in physics, in a work published around 1696. But by then, Newton had fallen into the “vis viva” trap. After two extensive revisions of the book by Newton and others, the “vis viva” remains clearly visible, though artfully disguised, in the Principia.

To clearly see why Newton did what he did with the vis viva, it is also useful to quickly review how the concept came about, in the first place.

The inspiration for the “vis viva,” the forerunner of the modern kinetic energy, originated with Galileo. Galileo’s equations for uniformly accelerated motion include the following equations, which are familiar to everyone,

$$(1) \quad v = at$$

$$(2) \quad h = \frac{1}{2} gt^2$$

$$(3) \quad v^2 = 2gh$$

Where, v is velocity, a , or g is acceleration, t is time, and h is the height in free fall.

Leibniz began with eq. 3, which is used today to calculate the velocity of a descending stone, roller coaster, or space probe; or a cannon ball projected upwards. He multiplied both sides of the equation by the mass, m , a valid mathematical step, to obtain,

$$(4) \quad 2mgh = mv^2$$

The present energy equation, i.e., potential energy = kinetic energy, can be derived from the last expression, by dividing both sides by 2 to obtain,

$$(5) \quad mgh = \frac{1}{2} mv^2$$

Leibniz was specific about the value of the vis viva, i.e., mv^2 . He referred to the expression “ $2mgh$ ” as the force needed to lift a body a distance h , or the weight times the height. The author is not aware of how Leibniz handled the factor “2” on the left in eq. 4, or the factor “ $\frac{1}{2}$ ” on the right in eq. 5. The same issue applies to Newton, as will be described later. Setting aside the factors 2 and $\frac{1}{2}$, Leibniz referred to both energy expressions in eq. 4 by the word force.

The “vis viva” was hotly debated for 200 years after it was introduced. In the derivation of the law of conservation of energy in the 19th century, Young, Carnot, Mayer, Joule, Helmholtz and others used the terms “force” and “vis viva” for energy.

The Principia was controversial from the beginning. When the book was first published, Leibniz and Newton’s own countryman, Robert Hooke, accused Newton of infringing on their work. Newton canceled all reference to Leibniz and demoted the great contributions of Hooke in the book. These events have been widely covered in scientific and popular literature in the last 300 years, but, the vis viva controversy was never resolved.

Dispassionate examination of the Principia shows that Newton did a strange thing in the book. Specifically, Newton deleted the mass, m , from the expression mv^2 , and he called the remaining term “ v^2 ” motion. In particular, Newton opted to use Galileo’s eq. 3, or $v^2 = 2gh$, in place of Leibniz’s equation, $mv^2 = 2mgh$, for the Second Law of Motion. No one noticed Newton’s choices for 300 years. The choices made for awkward writing by Newton, from which he was unable to recover for forty years after the initial publication of the Principia.

The author has carefully selected some Examples from the Principia to corroborate the thesis of this paper. Special care is taken to make the examples clear to everyone.

EXAMPLE 1

Historians of science have expressed astonishment that Newton did not credit Descartes for the “motion” used in the Second Law; instead, Newton credited Galileo. Specifically, Newton writes in the Scholium appended to the Laws of Motion,

“By the first two Laws ... Galileo discovered that the descent of bodies observed the duplicate ratio of the time ...”

We all know how Galileo was relevant to the First Law of Motion, or the Law of Inertia. But, how was Galileo relevant to the Second Law? How was Galileo relevant to $F=ma$? This has been considered a mystery. The mystery disappears when one realizes that Newton's Second Law is strictly a statement of Galileo's equation $v^2 = 2gh$, as clearly shown below.

EXAMPLE 2

The confusion surrounding the issue arises from Newton's arbitrary use of the word "force," and his proclivity to needlessly complicate elementary matters. Newton defines "force," before the Laws of Motion, as originating "from percussion, from pressure, from centripetal force," and the centripetal force to consist of "three kinds: absolute, accelerative, and motive." Setting these and other complications aside, let us look at how Newton uses the word "force."

Consider the widely known equation of distance, or space, in free fall, or eq. 2, $h = \frac{1}{2} gt^2$. The equation can be written in three forms, using elementary algebra, as follows,

$$(a) \quad h = \frac{1}{2} gt^2$$

$$(b) \quad g = 2h/t^2$$

$$(c) \quad t^2 = 2h/g$$

If we ask a modern mathematician, scientist, engineer or philosopher to describe the above relationships in Euclidean (or Newtonian) language, he or she might write, for (a) the space is proportional to the acceleration and the squares of the times directly; for (b) the acceleration is proportional to the spaces directly and the squares of the times inversely, and for (c) the squares of the times are as the spaces directly, and the acceleration inversely.

Let us now look at how Newton describes the above three forms of the simple free fall equation. The reader can find this in the Corollaries to LEMMA X of Book I in the Principia.

For $h = \frac{1}{2} gt^2$, Newton writes, "*COR. 3. ... any spaces whatsoever ... are as the forces and the squares of the times conjunctly.*"

For $g = 2h/t^2$, Newton writes, "*COR. 4. And therefore the forces are as the spaces ... directly, and the squares of the times inversely.*"

And for $t^2 = 2h/g$, Newton writes, "*COR. 5. And the squares of the times are as the spaces described directly, and the forces inversely.*"

Note that Newton specifically calls the acceleration “g” “forces.” Leibniz called mgh force; Newton calls g force. We call g acceleration; Newton calls it force.

In Corollary 1 of the same Lemma X, Newton refers to “g,” as “*forces ... applied to the bodies.*” The mass, m , is not mentioned, implicitly or explicitly, by Newton.

As elsewhere in the Principia, Newton ignores the factor “ $\frac{1}{2}$ ” in Corollary 3, and the factor “2” in Corollaries 4 and 5. The same was mentioned of Leibniz earlier. These factors are more important in mathematics and physics than has been realized to date. Principia scholars should reexamine how Newton, and others, handled the factors “2” and “ $\frac{1}{2}$ ” in their work.

EXAMPLE 3

The reader should reexamine the Principia objectively, and not be intimidated by the seemingly indecipherable book. The book is written in simple Euclidean form that is needlessly, though deliberately, complicated by Newton. The following is a conspicuous, easily verifiable, example of how Newton represents Leibniz’s equation, $2mgh = mv^2$, geometrically, but uses Galileo’s equation, $v^2 = 2gh$, verbally. This Example alone should leave no doubt in anyone’s mind about the true meaning of Newton’s Second Law of Motion. All educators must grasp this Example to explain its import to their students.

The problem of a body, of weight mg , falling freely from a given height, h , or projected upwards to the same height, is found in all introductory physics textbooks, including, high school books. This is the simplest free fall problem involving force and distance. The force, mg , and the distance, h , can be represented as shown in Fig. 1. The area of the typical resulting rectangle is equal to the work (force times distance, or mgh) or the potential energy (mgh). How is the area of the rectangle, i.e., the potential energy, related to the maximum kinetic energy? One can use simple algebra to show that the area of mgh is equal to the area of $\frac{1}{2}mv^2$, by substituting for v^2 from Galileo’s equations, or,

$$mgh = \frac{1}{2} mv^2$$

$$mgh = \frac{1}{2} m(2gh)$$

$$mgh = mgh$$

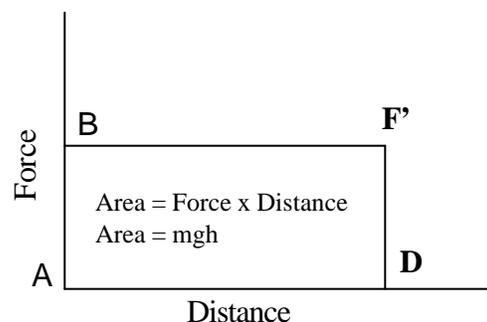


Fig. 1 Force vs. Distance in Local Free Fall

And so, the area of the rectangle representing the potential energy, mgh , is equal to the area representing the kinetic energy. This type of problem is not treated in physics books, primarily because, 100 years ago, Hertz, Mach, Boltzmann and others were unable to incorporate the force (ma or mg) into the energy relationships. This limitation is solved in future papers.

How does Newton represent the above simple free fall problem, schematically and verbally?

Fig. 2 is Newton's geometrical representation of the above simple problem. The force is shown on the horizontal axis, and the height is shown on the vertical axis. Because locally, the force, $F=ma$, is constant, the potential energy will be represented by the simple rectangle $ABF'D$, where F' is added to Newton's original diagram.

Newton complicates matters by showing the centripetal force to vary with height, the curved line BFG . This is true for the distance to the Moon, but in local free fall, the force is constant. Newton knows this and so, for local free fall, he writes in Proposition 39, Problem 27, Book I of the Principia, "suppose EG to coincide with the perpendicular AB ." He now obtains the rectangle $ABF'D$, which is similar to our rectangle in Fig. 1. Although the simple rectangle is distinctly described verbally by Newton, he does not show it schematically.

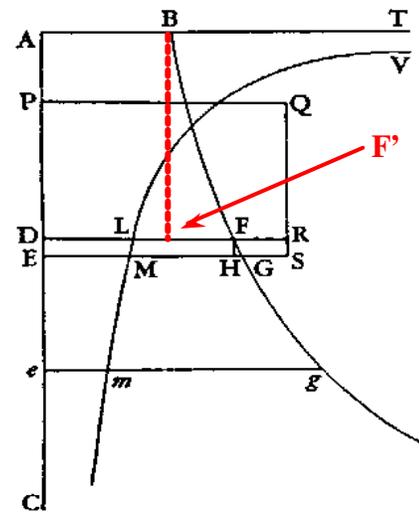


Fig. 2 Newton's Schematic for Free Fall
Red dotted line illustrates Newton's words.
For local free fall, compare $ABF'D$ to Fig. 1.
Source: Principia, Book 1, Proposition 39

After elaborate analysis, Newton concludes (in paragraph 3) that, "the area itself will be as the square of that velocity," or v^2 . Referring to his diagram, or Fig. 2, Newton writes, "the area $ABFD$ will be as VV ," or v^2 . The previous simple algebraic steps show this instantly.

Newton demonstrates here that the area, we call mgh , is proportional to " v^2 ." Do you see the mess? If Newton's "force," on the horizontal axis, is the familiar $F=ma$, or $F=mg$, then the area of the rectangle would be "as" mv^2 , or the vis viva, and not "as VV ," as Newton writes.

There should be no question about it. Newton's Second Law of Motion, as he demonstrates in Proposition 39 and elsewhere, is Galileo's equation $v^2 = 2gh$. This explains why Newton credited Galileo for the Second Law of Motion, which is not related to the equation $F=ma$ in any way.

The author will add that Newton's elaborate infinitesimal analysis for local free fall, as given in Proposition 39, is itself wrong. This will be explained in future papers and lectures. Physicists, mathematicians and engineers, and, even, students, can now step up to the challenge and try to find out what is wrong with Newton's infinitesimal analysis and, more importantly, to develop the correct mathematical and physical solutions.

EXAMPLE 4

The reverse of the Second Law of Motion is given by Newton in Definition VIII at the beginning of the Principia. In the Second Law, "the alteration of motion," Newton writes, is proportional to the "motive force impressed." Here, the motive force, he writes, is "proportional to the motion." How can this be? How can this be overlooked for 300 years? The hallmark of the force, in the equation $F=ma$, is that it is constant. In free fall, the moving body accelerates, its momentum (mv) increases and its energy (mv^2) increases; but the "force" that causes the acceleration is constant. Newton knows this. Although he does not show the constant force, $F=ma$, in his Fig. 2 in the previous Example, he says it in distinct words in Proposition 39, "*suppose EG to coincide with the perpendicular AB,*" for local free fall.

The force in $F=ma$ cannot be proportional to the motion, whether the motion is Descartes' or Leibniz's. Newton's "motive force" in the Second Law is not the force $F=ma$. The reader will note that the potential energy, mgh , is proportional to mv^2 , and $2gh$ is proportional to v^2 . This is another distinct proof that Newton's Second Law is Galileo's equation, $v^2 = 2gh$.

EXAMPLE 5

Newton defined important terms before he used them, but he did not define the word "alteration" before the Second Law. Examination of the Second Law and Definition VIII, together, reveals that the word "alteration," in the Second Law, is used for its plain meaning in the dictionary; e.g., to change a thing without changing its nature. If $1a = 1b$, then, $2a = 2b$, $3a = 3b$, and so on. The reader can hear this concept in Newton's brief explanation of the Second Law,

"If any force generates a motion, a double force will generate double the motion, a triple force triple the motion ..."

In Galileo's equation and Newton's Second Law, or, $v^2 = 2gh$, the left and right sides are proportional. If the value of one is "altered," e.g., doubled or tripled, the value of the other side is

proportionately altered. In Newton's example, or $2gh = v^2$, we have $2 \times 2gh = 2 \times v^2$, $3 \times 2gh = 3 \times v^2$, and so on. There is no hint here of any fluxional or differential analysis.

The widely held belief that Newton's word "alteration" in the Second Law means the "rate of change," or, the "derivative," is wrong. As amply shown above, Newton's "motion" is v^2 , which he uses in place of mv^2 . Furthermore, the derivative of either v^2 or mv^2 has nothing whatsoever to do with the equation $F=ma$, as all teachers of mathematics can show.

EXAMPLE 6

The Holy Grail for Newton's Principia is universal gravitation. Newton does not claim that he discovered any of the Three Laws of Motion. The title for the Laws states "Axioms, or Laws of Motion." Axioms are a priori known facts. According to Newton, the Laws of Motion were axiomatic, known and accepted before his book was written.

In the Scholium to the Laws of Motion, Newton writes,

"Hitherto I have laid down such principles as have been received by mathematicians, and are confirmed by abundance of experiments."

Newton expressly writes that he (a mathematician among mathematicians) "received" the Laws. From whom were the Laws received? Immediately after this sentence, Newton credits Galileo for the First and the Second Laws, as described earlier. Who delivered the Third Law of Motion, or the Action-Reaction Law? It has not been recognized before, but Aristotle stated the Action-Reaction Law verbatim two thousand years before the Principia. In his distinctive style, Aristotle writes, "*As the Pusher pushes, so is the Pushed push, and with equal force,*" in *On the Motion of Animals*. Newton gives the following example for the Third Law, "*If you press a stone with your finger, the finger is also pressed by the stone.*" Here, the finger is the Pusher, and the stone is the Pushed. Newton's Third Law of Motion is evidently Aristotle's Action-Reaction principle.

Newton's Second Law of Motion is a vague statement of Galileo's equation, $v^2 = 2gh$, which Newton might have used to distinguish it from Leibniz's energy equation, $mv^2 = 2mgh$. There is one difference between the two forms, in Newton's, the mass, m , is deleted.

Author's Note

The Principia has been described as the greatest scientific work ever. Yet, some say that, when published, only two people could understand it, Huygens and Leibniz. Others say that 5 or 7

people could understand the book then, including, Hooke. But, Huygens, Hooke and Leibniz criticized the book. Cambridge scholars, who received copies of the book when first published, said that they could spend seven years studying it before they understood a word of it. In spite of this, the book is the basis of design of important modern systems.

Newton declared that he wanted the book to be undecipherable. The book is decipherable, as can be seen from the evident Examples given above. The legacy of the Principia will be revised.

Rather than open up doors of opportunity, especially, in mathematics and physics, the Principia impeded progress, particularly, in England. In early 19th century, great English scientists, from Cambridge and elsewhere, lamented the decline of the sciences in England, for the hundred years after the Principia. Afterwards, electromagnetic and quantum theories, which do not rely on Newton's book, and their associated technologies, advanced in leaps and bounds in the 20th century. In mechanical and aerospace engineering, and the author is a mechanical/aerospace engineer, progress has been tortuous, and problems have been many and expensive. Here, the primary tools of the aerospace engineer include Newton's Three Laws of Motion, which, as shown above, are not Newton's, and the most important tool is the equation $F=ma$, which, as also shown above, is not Newton's Second Law of Motion.

Aerospace systems are the ultimate test of Newtonian mechanics. The complex mathematical analysis of the 1960's, augmented, for the first time, with mainframe computer capabilities, led us to phenomenal achievements, but also costly failures. We designed complex systems. We tested the systems, subsystems and components. When the first tests failed, we redesigned and retested. A perilous trend was born. Elaborate and expensive tests became the tool to arrive at workable systems. In early 1970's, senior managers at COMSAT gave the author free hand to research the subject of this paper and to institute corrections in spacecraft and terrestrial systems. Despite numerous successes in the 1970s, the task was difficult. In the next decade, senior managers from NASA recommended that the space agency look into the matter. No one dared question the dogmatic belief in the Principia.

The author's attempts to correct the above destructive trend are widely known in public, private and academic circles, especially, in the aerospace communities. The intransigence of aerospace experts in this matter has brought the space program to its knees. As widely reported today, even the top experts do not believe in the possibility of safe spaceflight. Yet, very safe economical

spaceflight can be achieved. The first step is that every engineer, scientist and mathematician must recognize what the equation $F=ma$ means and what it does not mean, who derived it and who didn't, and how to use the equation correctly everywhere it is used.

Examples 1-6 clearly show that the equation $F=ma$ is not embodied in Newton's Second Law of Motion. Furthermore, it is shown that the governing equation in the Second Law is the simple free fall equation, $v^2=2gh$, which is found in all introductory physics textbooks. If Newton did not develop the equation $F=ma$, then, who did?

The Original Derivation of the Equation $F=ma$

Some may think it plausible that any of the 18th century mathematicians, who used the calculus after Newton and Leibniz, might have been the first to develop the equation, e.g., the Bernoullis, Euler, D'Alembert, Laplace, Lagrange, etc. But that is not the case.

Galileo's student, Evangelista Torricelli, correctly derived the equation before the time of Newton and Leibniz. Here is a quick reconstruction of how Torricelli did it.

Torricelli began with Galileo's elementary equation of uniformly accelerated motion, or,

$$(1) \quad v = at$$

Multiplying both sides of the equation by the mass, m , of a moving body, a valid mathematical step, Torricelli obtained,

$$(6) \quad mv = mat$$

For the very familiar "ma" on the right side of the equation, Torricelli substituted the force, F . Switching terms, the equation becomes,

$$(7) \quad Ft = mv$$

This is the impulse equation that we use today to calculate the force on, or velocity of, a golf ball struck with a club or a baseball hit with a bat; where the time of contact is brief. Dividing both sides of the equation by the time, t , Torricelli obtained,

$$(8) \quad F = m (v/t)$$

But, v/t is equal to the acceleration, a , from Galileo's eq. 1. Substituting for v/t , we obtain,

$$(9) \quad F = ma$$

In the last four steps, the important equation $F=ma$ is derived twice, using elementary algebra, and without any calculus.

Torricelli explained the meaning of the equation $F=ma$ with remarkable examples, for his time, in the cases of free fall and uniformly accelerated horizontal motion. He also noted that when the time, t , in eq. 8 approaches zero, the force becomes infinite. To sidestep the mathematical difficulty, Torricelli used infinitesimal units of time in his analysis to become one of the precursors of the calculus, with Descartes, Fermat, Pascal, Borelli, Wallis and Isaac Barrow.

Conclusion

This paper is part of Lecture One, of Four Lectures, proposed by the author in May 2005, with the titles, (1) $F=ma$, Important Equation, Big Mistake, (2) Finding the Lost Force and A New Principle of Superposition, (3) True Form and Meaning of the Conservation of Energy and Momentum, and (4) Mathematics of Dialectic and the Forms. The Lectures identify big mistakes, and present well thought out solutions, in mathematics, physics, engineering and philosophy. The mistakes continue to adversely affect important subjects and systems.

In this paper, it is shown that the very important equation $F=ma$ is not embodied in Newton's Second Law of Motion. It is also shown that Newton's Second Law is an odd representation of the well-known equation of free fall, or $v^2 = 2gh$. The original straightforward derivation of the equation $F=ma$ by Torricelli, which can be easily followed by experts and students alike, is also described.

When faced with the difficult choice between truth and dear friends, Aristotle chose Truth over a very dear and great friend, Plato. Today, the reader faces the same choice between the Truth and another great friend, Sir Isaac Newton. Similar difficult choices await the reader in the remaining Lectures and papers.

The equation $F=ma$ is at the foundation of modern physics and engineering. The equation is not an end; it is a means to a surprising end. It seems that only Galileo tried to achieve the important end in the last 400 years. Experts and students can begin the quest for the surprising end, by trying to find answers to the questions raised in this, and future, papers. The author first detected errors in the Principia while a student in high school in Amman, Jordan in the 1950's. Excellent

education and mentoring played a key role then. Educators and mentors should not hesitate to involve young students in this intellectual quest.

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